# SEQUENCES

Comments and extensions to the presentation on sequences. Arranged by titles of individual slides.

1. **Sequences. Notation and examples**.

   As sequences are functions from **N** into **R** we can denote the set of all sequences by $\mathbf{R}^{\mathbf{N}}$. You may remember our discussion of $Y^X$ denoting the set of all functions from X into Y, when we said that the notation nicely reflects the fact that the number of functions from an n-element set X into a k-element set Y is $k^n$, i.e. $|Y^X| = |Y|^{|X|}$. On the other hand, sequences can be considered "ordered sets" of numbers hence the notation $(a_n)$ seems more appropriate than $\{a_n\}$ by analogy with (a,b) and {a,b} denoting, respectively, ordered and unordered pairs.

   The set $\mathbf{R}^{\mathbf{N}}$ is an algebra with respect to addition and multiplication. What is the sum (product) of two sequences, you ask? Well, sequences are functions, we add functions by values, i.e. (f+g)(x) is defined as f(x)+g(x), hence $(a_n)+(b_n) = (a_n+b_n)$. The same for multiplication. For example, let $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. Then the sequence $(a_n)+(b_n) = (a_n+b_n) = (0,0,0,0, \dots)$ is the sequence of infinitely many zeroes, a *constant* sequence.

   *Question.* What are algebraic properties of the algebra $(\mathbf{R}^{\mathbf{N}},+,\cdot)$? Is it a field?

2. **Arithmetic sequence**.

   **Definition** *A sequence $(a_n)$ is called an <u>arithmetic sequence</u> iff* $(\exists d \in R)(\forall n \in N) a_{n+1} - a_n = d$.

   The number $a_1$ is then called the initial or starting value, $d$ is referred to as *the increment* or the *difference*. Let us calculate the sum of the first $n$ terms of an arithmetic sequence.

   $a_1+a_2+ \dots + a_n = a_1+(a_1+d) + \dots + (a_1 +(n-1)d) = na_1+(0+d+2d+ \dots +(n-1)d) = na_1 + d(0+1+2+ \dots n-1)$.

   There is a well-known trick (allegedly invented by Gauss) for doing the sum $0+1+2+ \dots n-1$. Namely one should notice that $0+n-1$ is the same as $1+n-2$, and $2+n-3$ and each is equal to $n-1$. Since the number of these pairs is $\frac{n}{2}$ the total is $\frac{n(n-1)}{2}$, hence the answer is $na_1 + d\frac{n(n-1)}{2}$.

3. **Geometric sequence**.

   **Definiton** *A sequence $(a_n)$ is called a <u>geometric sequence</u> iff*

   $(\exists q \in R)(\forall n \in N)(a_{n+1} = a_n q \ \&\ q \neq 0)$. ("$q$" for quotient, obviously).

   In the definition we do not say "$\frac{a_{n+1}}{a_n}$ is constant" because that would exclude the constant sequence (0,0,…).

   On the other hand the conditon $q \neq 0$ excludes sequences like (2,0,0,0,..) (which is good) while including (0,0,…) where $q$ can be any nonzero – which is good, too.

   Let us now calculate the sum of the first $n$ terms of a geometric sequence.

   $S_n = a_1+a_2+ \dots + a_n = a_1+qa_1+ \dots + q^{n-1}a_1 = a_1(1+q+ \dots +q^{n-1})$ which means we must develop a method for doing the sum $1+q+ \dots +q^{n-1} = T_n$. Multiplying both sides by $1-q$ we get $(1+q+ \dots +q^{n-1})(1-q) = T_n(1-q)$ which yields $1+q+ \dots +q^{n-1} - q - \dots - q^n = T_n(1-q)$ and, consequently, $1-q^n = T_n(1-q)$. Hence $S_n = a_1 \frac{1-q^n}{1-q}$. As you see the formula is meaningless for $q$=1. In this case the sequence $(a_n)$ is constant and $S_n = na_1$. The formula does make sense, though, for $q = -1$. In this case $S_1 = S_3 = S_5 = \dots = a_1$ and $S_2 = S_4 = S_6 = \dots = 0$.

4. **Monotonic sequences**.

   Just remember that "non-increasing" is NOT the negation of "increasing" but something that you would perhaps like to call "almost decreasing". The same applies to "non-decreasing". Thus every constant sequence is both non-increasing and non-decreasing while it is neither increasing nor decreasing. This might be confusing since, for example, "non-empty" does mean the opposition to "empty". Here, alas, no.

5. **Bounded sequences**. I have nothing to add here.

6. **Subsequences**.

   This subject appears in the *Limits: properties* section of the presentation on sequences but, since it is often confusing for students, I decided to include it here, in the *Sequences: general notions* section.

**Definition**. If $(a_n)$ is a sequence then for every increasing sequence of natural numbers $(k_n)$ the sequence $(a_{k_n})_{n=1}^{\infty}$ is called a *subsequence* of $(a_n)$.

Essentially this means that a subsequence of $(a_n)$ is a sequence (hence infinite) resulting from the removal from $(a_n)$ some (possibly none) of its terms without changing the order of the remaining terms

**Examples**. Every sequence is its own subsequence.

(1,2,3,4) is not a subsequence of $(n)_{n=1}^{\infty}$ because it is finite.

(3,1,5,7,9,11, …etc.) is not a subsequence of $(n)_{n=1}^{\infty}$ even though odd positive integers do form a subset of **N** but the order of terms is messed-up.

(1,3,5,7,9, … etc.) corresponding to the $k_n = 2n-1$ increasing sequence of naturals.

$(p_n)$ = (2,3,5,7,11,13, … (*all primes*)) is a subsequence of $(n)_{n=1}^{\infty}$ even though it is not possible to directly define the sequence of subscripts (other than by saying "take consecutive primes").

$(a_{p_n})$ is the subsequence of $(a_n)$ consisting of all terms whose subscripts are primes.

The sequence $(a_{2n})_{n=1}^{\infty}$ is the subsequence of $(a_n)$ consisting of all even-subscripted terms of $(a_n)$.

## Comprehension test.

1. Prove or disprove:
   a) Every subsequence of an increasing (decreasing, non-increasing, non-decreasing) is increasing (decreasing, etc.).
   b) Every sequence contains a subsequence which is non-decreasing or non-increasing.
   c) Every sequence contains a subsequence which is decreasing or increasing.
   d) Every sequence contains at least one non-increasing and at least one non-decreasing subsequence.
   e) Every sequence which contains at least one non-increasing and at least one non-decreasing subsequence is constant.
2. Prove or disprove:
   a) Every subsequence of a bounded sequence is bounded.
   b) No monotonic sequence is bounded.
   c) Every sequence contains a bounded subsequence.
   d) The sum (product) of two monotonic sequences is monotonic.
   e) The sum (product) of two bounded sequences is bounded.

# LIMIT OF A SEQUENCE

## 1. Limits – definition.

Not much to add here. Just a warning. This definition is complicated. You must not only memorize it but understand it.

The answer to the obvious question "Does it have to be this complicated?" is YES. Students usually complain that it should be enough to say "The larger $n$ the closer we get to the limit L". They are wrong (obviously). Such a definition would only cover those cases, where the distances between $a_n$ and L form a decreasing sequence which obviously does not take care of many sequences which intuitively should be considered convergent to some number. And this will be your comprehension test:

Construct a sequence $(a_n)$ convergent to 0 and such that for infinitely many natural numbers $k$ $|a_{k+1}| > |a_k|$ - which means the distance from 0 to $a_{k+1}$ is actually greater than from 0 to $a_k$.